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A new parametric method for ranking fuzzy numbers

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Abstract

Ranking fuzzy numbers is important in decision-making, data analysis, artificial intelligence, economic systems and operations research. In this paper, to overcome the limitations of the existing studies and simplify the computational procedures an approach to ranking fuzzy numbers based on α-cuts is proposed. The approach is illustrated by numerical examples, showing that it overcomes several shortcomings such as the indiscriminative and counterintuitive behavior of existing fuzzy ranking approaches.

Keywords: Fuzzy number; Ranking fuzzy numbers; α-cuts

1. Introduction

Fuzzy numbers are an important issue in research in fuzzy set theory [16]. Because of the suitability for representing uncertain values, fuzzy numbers have been widely used in many applications and various methods manipulating them have also been developed [2,4,8,26,28]. The results of studies on ranking fuzzy numbers have been used in application areas such as decision-making, data analysis, artificial intelligence and socioeconomic systems. In order to rank fuzzy numbers, one fuzzy number needs to be evaluated and compared to the others, but this may not be easy. Since fuzzy numbers are represented by possibility distributions, they can overlap with each other and, thus, it is difficult to determine clearly whether one fuzzy number is larger or smaller than another.

In the literature about fuzzy sets, ranking fuzzy numbers is well investigated because of its widespread usage in the area of decision making. It is a necessity to rank the obtained fuzzy

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numbers in a decision-making problem. The ranking methods can be classified in three categories. The first category directly transforms each fuzzy number into a crisp real number and the second category compares a fuzzy number to all the other $n - 1$ fuzzy numbers to obtain its mapping into a positive real number. The third category differs substantially from the first two. In this category, a method for pairwise ranking or preference for all pairs of fuzzy numbers is determined and then based on these pairwise orderings, a final order of the n fuzzy numbers is attempted. Two factors play significant roles in fuzzy decision systems:

(1) Contribution of the decision-maker in the decision making process,
(2) Simplicity of calculation.

The proposed method in the present paper has both of the factors. The first factor, because the choice of α depends on the decision maker and it plays an important role in this paper.

The significance of ranking fuzzy numbers for solving real world decision problems in a fuzzy environment [27] has led to tremendous efforts being spent on the development of various ranking approaches [1,3,2,4–10,12,11,13–30]. These approaches can be categorized into mathematical approaches and linguistic approaches. Linguistic approaches focus on the development and the use of linguistic terms for describing the ranking outcome which is not sequential. This study presents a mathematical approach to ranking the fuzzy numbers.

More than 30 fuzzy ranking indices have been proposed since 1976. Jain [14,15], Dubois and Prade [11] introduced the relevant concepts of fuzzy numbers. Bortolan and Degani [1] reviewed some methods to rank fuzzy numbers, Chen and Hwang [3] proposed fuzzy multiple attribute decision making, Choobineh and Li [6] proposed an index for ordering fuzzy numbers, Dias [10] ranked alternatives by ordering fuzzy numbers, Lee and Lee-Kwang [19] ranked fuzzy numbers with a satisfaction function, Requena et al. [24] utilized artificial neural networks for the automatic ranking of fuzzy numbers, Fortemps and Roubens [13] presented ranking and defuzzification methods based on area compensation, and Raj and Kumar [23] investigated maximizing and minimizing sets to rank fuzzy alternatives with fuzzy weights. Chu and Tsao [7] proposed a method of ranking fuzzy numbers with an area between the centroid and original points. Chu and Tsao's method originated in the concepts of Lee, Li [20] and Cheng [5]. Lee and Li proposed the comparison of fuzzy numbers, for which they considered mean and standard deviation values for fuzzy numbers based on the uniform and proportional probability distributions. Wang and Lee [26] presented a new method of ranking fuzzy numbers using radius of gyration. In this paper, we shall propose a new method for ranking fuzzy numbers to overcome the shortcomings of some previous methods.

It should be noted that many existing fuzzy number ranking methods tried to make a comparison of the fuzzy numbers in an objective way. However, an important aspect of the fuzzy number applications is that it can represent the subjective knowledge of the decision maker. Since the results of comparison in real problems affect implicated individuals, the decision maker's subjective attitude should be reflected in the process of ranking. However, the objective ranking methods use a neutral attitude to evaluating them. In previous research, some preference methods were suggested. However, most of them were rather simple. They often only considered two extremes (optimistic and pessimistic) and just linearly combined the results with both the extremes. On the other hand, the method proposed in this paper, can represent the decision maker's preference information explicitly.

Our discussion is presented in 5 sections. In Section 2, we give some definitions and preliminaries. In Section 3, we describe the proposed method. In Section 4, we first give some examples to state the shortcomings of previous methods and then present a new method to overcome these problems. Moreover, we shall compare our method with some ranking methods of fuzzy numbers. Finally, the conclusion is given in Section 5.

2. Preliminaries

A review of some basic notions of fuzzy sets is presented. These notions are expressed as follows:

Definition 2.1 ([1]). A fuzzy number is a fuzzy set like $u : R \to I = [0, 1]$ which satisfies:

(1) u is upper semi-continuous,
(2) $u(x) = 0$ outside some interval $[a, d]$,
(3) There are real numbers b, c such that $a \le b \le c \le d$

and

(a) $u(x)$ is monotonic increasing on $[a, b]$,
(b) $u(x)$ is monotonic decreasing on $[c, d]$,
(c) $u(x) = 1, b \le x \le c$.

The membership function u can be expressed as

$$u(x) = \begin{cases} u_L(x), & a \le x \le b, \\ 1, & b \le x \le c, \\ u_R(x) & c \le x \le d, \\ 0, & \text{otherwise} \end{cases}$$

where $u_L : [a, b] \to [0, 1]$ and $u_R : [c, d] \to [0, 1]$ are left and right membership functions of fuzzy number u. An equivalent parametric form that we use in this paper is as follows:

Definition 2.2. A fuzzy number $\tilde{u}$ in parametric form is an ordered pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le \omega$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function over $[0, \omega]$,
2. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function over $[0, \omega]$,
3. $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le \omega$.
   ω is an arbitrary constant such that $0 < \omega \le 1$. A crisp (non-fuzzy) number "k" is simply represented by $\underline{u}(r) = \overline{u}(r) = k,\ 0 \le r \le \omega$.

By Definition 2.2, the fuzzy number space $\{\underline{u}(r), \overline{u}(r)\}$ becomes a convex cone E^1 which is embedded isomorphically and isometrically in a Banach space. If $\tilde{u}$ is an arbitrary fuzzy number then the α-cut of $\tilde{u}$ is $\left[\tilde{u}\right]_\alpha = \left[\underline{u}(\alpha), \overline{u}(\alpha)\right],\ 0 \le \alpha \le \omega$.

If $\omega = 1$, then the above-defined number is called a normal fuzzy number. Here $\tilde{u}_\omega$ represents a fuzzy number in which "ω" is the maximum membership value that a fuzzy number takes on. Whenever a normal fuzzy number is meant, the fuzzy number is shown by $\tilde{u}$, for convenience. A generalized trapezoidal fuzzy number $\tilde{u}$ of the universe of discourse X can be characterized by a trapezoidal membership function parameterized as (a, b, c, d, ω), where $0 < \omega \le 1$ and a, b, c and d are real values. If $\omega = 1$, we have a normal trapezoidal fuzzy number that it is denoted as $\tilde{u} = (a, b, c, d, 1)$.

We can see that if $a = b$, $c = d$ and $\omega = 1$, then $\tilde{u}$ is called a crisp interval; if $a = b = c = d$ and $\omega = 1$, then $\tilde{u}$ is a crisp value. If $b = c$, then $\tilde{u}$ becomes a generalized triangular fuzzy number, and it can be parameterized by (a, b, b, d, ω).

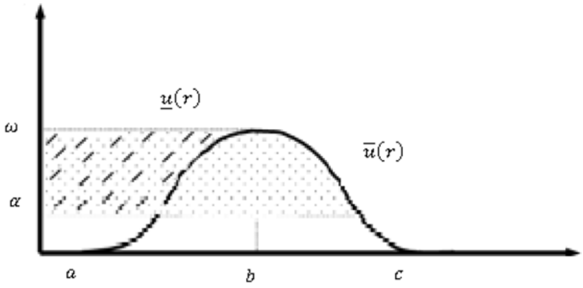

Fig. 1. An arbitrary fuzzy number.

3. Our method

Let $\tilde{u}_\omega = \left(\underline{u}(r), \overline{u}(r)\right)$, $(0 \le r \le \omega)$ be a fuzzy number based on the definition as mentioned of a fuzzy number. The value $Q_\alpha(\tilde{u}_\omega)$ is assigned to $\tilde{u}_\omega$ for a decision level higher than "α" which is calculated as follows:

$$Q_\alpha\left(\tilde{u}_\omega\right) = \int_\alpha^\omega (\underline{u}(r) + \overline{u}(r))dr, \quad 0 \le \alpha \le 1.$$

This quantity will be used as a basis for comparing fuzzy numbers in decision level higher than α.

We suppose, if $\alpha \ge \omega$, then $Q_\alpha\left(\tilde{u}_\omega\right) = 0$. In order to clarify the concept of the mentioned quantity, consider the following fuzzy number:

As shown in Fig. 1, the presented quantity is the summation of the cross-hatched area and the dotted area.

$$Q_\alpha\left(\tilde{u}_\omega\right) = \int_\alpha^\omega (\underline{u}(r) + \overline{u}(r))dr = \int_\alpha^\omega \underline{u}(r)dr + \int_\alpha^\omega \overline{u}(r)dr.$$

Definition 3.1. If $\widetilde{A}_\omega$ and $\widetilde{B}_{\omega'}$ are two arbitrary fuzzy numbers and $\omega, \omega' \in [0, 1]$, then we have:

1. $\widetilde{A}_\omega \preccurlyeq \widetilde{B}_{\omega'} \leftrightarrow \forall\alpha \in [0, 1] Q_\alpha\left(\widetilde{A}_\omega\right) \le Q_\alpha\left(\widetilde{B}_{\omega'}\right)$
2. $\widetilde{A}_\omega = \widetilde{B}_{\omega'} \leftrightarrow \forall\alpha \in [0, 1]\ Q_\alpha\left(\widetilde{A}_\omega\right) = Q_\alpha\left(\widetilde{B}_{\omega'}\right)$
3. $\widetilde{A}_\omega \succcurlyeq \widetilde{B}_{\omega'} \leftrightarrow \forall\alpha \in [0, 1]\ Q_\alpha\left(\widetilde{A}_\omega\right) \ge Q_\alpha\left(\widetilde{B}_{\omega'}\right)$.

Definition 3.2. If we compare two arbitrary fuzzy numbers including $\widetilde{A}_\omega$ and $\widetilde{B}_{\omega'}$ at decision levels higher than "α" and $\alpha, \omega, \omega' \in [0, 1]$, then we have:

1. $\widetilde{A}_\omega \preccurlyeq_\alpha \widetilde{B}_{\omega'} \leftrightarrow Q_\alpha\left(\widetilde{A}_\omega\right) \le Q_\alpha\left(\widetilde{B}_{\omega'}\right)$
2. $\widetilde{A}_\omega =_\alpha \widetilde{B}_{\omega'} \leftrightarrow Q_\alpha\left(\widetilde{A}_\omega\right) = Q_\alpha\left(\widetilde{B}_{\omega'}\right)$
3. $\widetilde{A}_\omega \succcurlyeq_\alpha \widetilde{B}_{\omega'} \leftrightarrow Q_\alpha\left(\widetilde{A}_\omega\right) \ge Q_\alpha\left(\widetilde{B}_{\omega'}\right)$

where $\widetilde{A}_\omega \preccurlyeq_\alpha \widetilde{B}_{\omega'}$, i.e., at decision levels higher than α, $\widetilde{B}_{\omega'}$ is greater than or equal to $\widetilde{A}_\omega$.

If α is close to 1, the pertaining decision is called a "high level decision", in which case only parts of the two fuzzy numbers, with membership values between "α" and "1", will be compared. Likewise, if "α" is close to 0, the pertaining decision is referred to as a "low level decision", since members with membership values lower than both the fuzzy numbers are involved in the comparison. For instance, as shown in Fig. 2, according to the presented quantity, the results clearly vary with different decision levels, e.g. $\widetilde{A} \preccurlyeq_{0.8} \widetilde{B}$, $\widetilde{A} \succcurlyeq_{0.1} \widetilde{B}$.

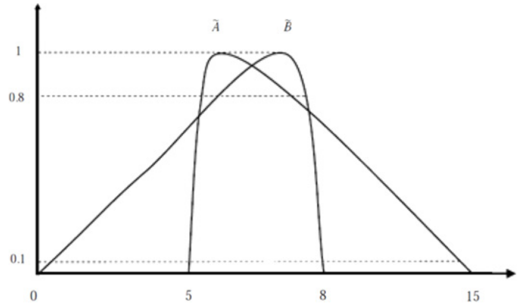

Fig. 2. Comparison of $\widetilde{A}$ and $\widetilde{B}$ at two different decision levels.

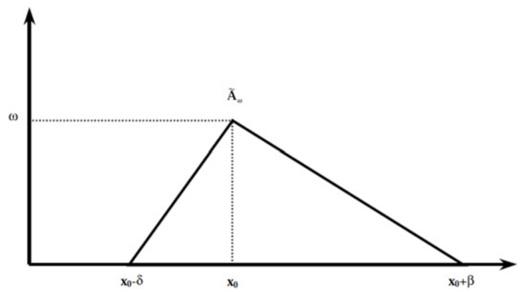

Fig. 3. A triangular fuzzy number.

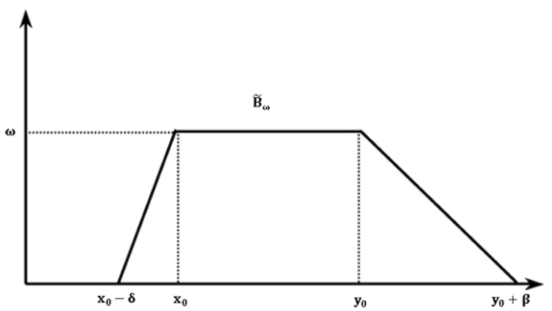

Fig. 4. A trapezoidal fuzzy number.

Two relevant classes of fuzzy numbers, which are frequently used in practical purposes and are rather easy to work with, are "triangular and trapezoidal fuzzy numbers".

As shown in Figs. 3 and 4, $\tilde{A}_\omega = \left(\underline{A}(r), \overline{A}(r)\right) = (x_0 - \delta + \frac{\delta}{\omega}r,\ x_0 + \beta - \frac{\beta}{\omega}r)$ and $\tilde{B}_\omega = \left(\underline{B}(r), \overline{B}(r)\right) = (x_0 - \delta + \frac{\delta}{\omega}r,\ y_0 + \beta - \frac{\beta}{\omega}r)$ are triangular and trapezoidal fuzzy numbers, respectively. The parametric values assigned to the two fuzzy numbers, represented by $Q_\alpha^{\mathrm{Tri}}\left(\widetilde{A}_\omega\right)$ and $Q_\alpha^{\mathrm{Tra}}\left(\widetilde{B}_\omega\right)$ respectively, may be calculated as follows:

If $\omega > \alpha$, then:

$$Q_\alpha^{\mathrm{Tri}}\left(\widetilde{A}_\omega\right) = \int_\alpha^\omega \{\underline{A}(r) + \overline{A}(r)\} dr$$
$$= \int_\alpha^\omega \left(2x_0 + (\beta - \delta)\left(1 - \frac{r}{\omega}\right)\right) dr = 2x_0(\omega - \alpha) + \frac{(\beta - \delta)}{2\omega}(\omega - \alpha)^2.$$

The value corresponding to the triangular fuzzy number $\widetilde{A}_\omega$ pertains to a higher level decision than α.

$$\begin{aligned} Q_\alpha^{\text{Tra}}\left(\widetilde{B}_\omega\right) &= \int_\alpha^\omega \{\underline{B}(r) + \overline{B}(r)\} dr \\ &= \int_\alpha^\omega \left(x_0 + y_0 + (\beta - \delta)\left(1 - \frac{r}{\omega}\right)\right) dr \\ &= (x_0 + y_0)(\omega - \alpha) + \frac{(\beta - \delta)}{2\omega}(\omega - \alpha)^2 \end{aligned}$$

where the value corresponding to the trapezoidal fuzzy number $\widetilde{B}_\omega$ pertains to a decision level higher than α.

Obviously, if $\alpha \geq \omega$, then the above quantity will be zero. It can also be seen that if $\tilde{A}$ is a normal triangular or trapezoidal fuzzy number ($\omega = 1$) the above quantities transform to:

$$\begin{aligned} Q_\alpha^{\text{Tri}}\left(\tilde{A}\right) &= 2x_0\,(1 - \alpha) + \frac{(\beta - \delta)}{2}\,(1 - \alpha)^2 \\ Q_\alpha^{\text{Tra}}\left(\widetilde{B}\right) &= (x_0 + y_0)\,(1 - \alpha) + \frac{(\beta - \delta)}{2}\,(1 - \alpha)^2, \quad 0 \leq \alpha \leq 1. \end{aligned}$$

As the above equalities show, if the fuzzy number is symmetrical ($\delta = \beta$), the equalities may be simplified more (the second terms on the right-hand side of the above equations are canceled out). For simplicity, based on the results obtained, hereafter the triangular fuzzy number $\tilde{A}$ and the trapezoidal fuzzy number $\tilde{B}$ will be represented as $\tilde{A} = (x_0, x_0, \delta, \beta, \omega)$ and $\tilde{B} = (x_0, y_0, \delta, \beta, \omega)$ respectively.

Remark. In the proposed method, the selection of α represents its flexibility. Flexibility is the ability to provide more than one index and/or allowing the participation of the decision makers. Unlike some other multiple index ranking methods, which leave the index selection task to the decision maker, the method of this paper lets the decision maker decide his/her valuation and optimistic level of confidence in the unit interval, and he/she will get the corresponding preference expectation of these fuzzy numbers in real number forms. For example, when the decision maker is risk-neutral, α near to 0.5 seems to be reasonable. A risk-averse decision maker might want to put α near to 1. For a risk-prone decision maker α near to 0 can be utilized. Similar works are presented in [9,21,25].

4. Numerical examples

In this section, three numerical examples are used to illustrate the proposed approach to ranking fuzzy numbers.

Example 1. Consider the data used in Ref. [4], i.e., eight sets of generalized fuzzy numbers as shown in Fig. 5, in order to compare the proposed method with Cheng's method (1998) [5], Chu and Tsao's method (2002) [7], Murakami et al.'s method (1983) [22], Yager's method (1978) [28], Chen and Chen's method (2009) [2] and Chen and Sanguansat's method (2010) [4]. The results are shown in Table 1. The results for the proposed method are calculated for two values of $\alpha, \alpha = 0.2, 0.8$, that correspond to a low level decision and a high level decision, respectively.

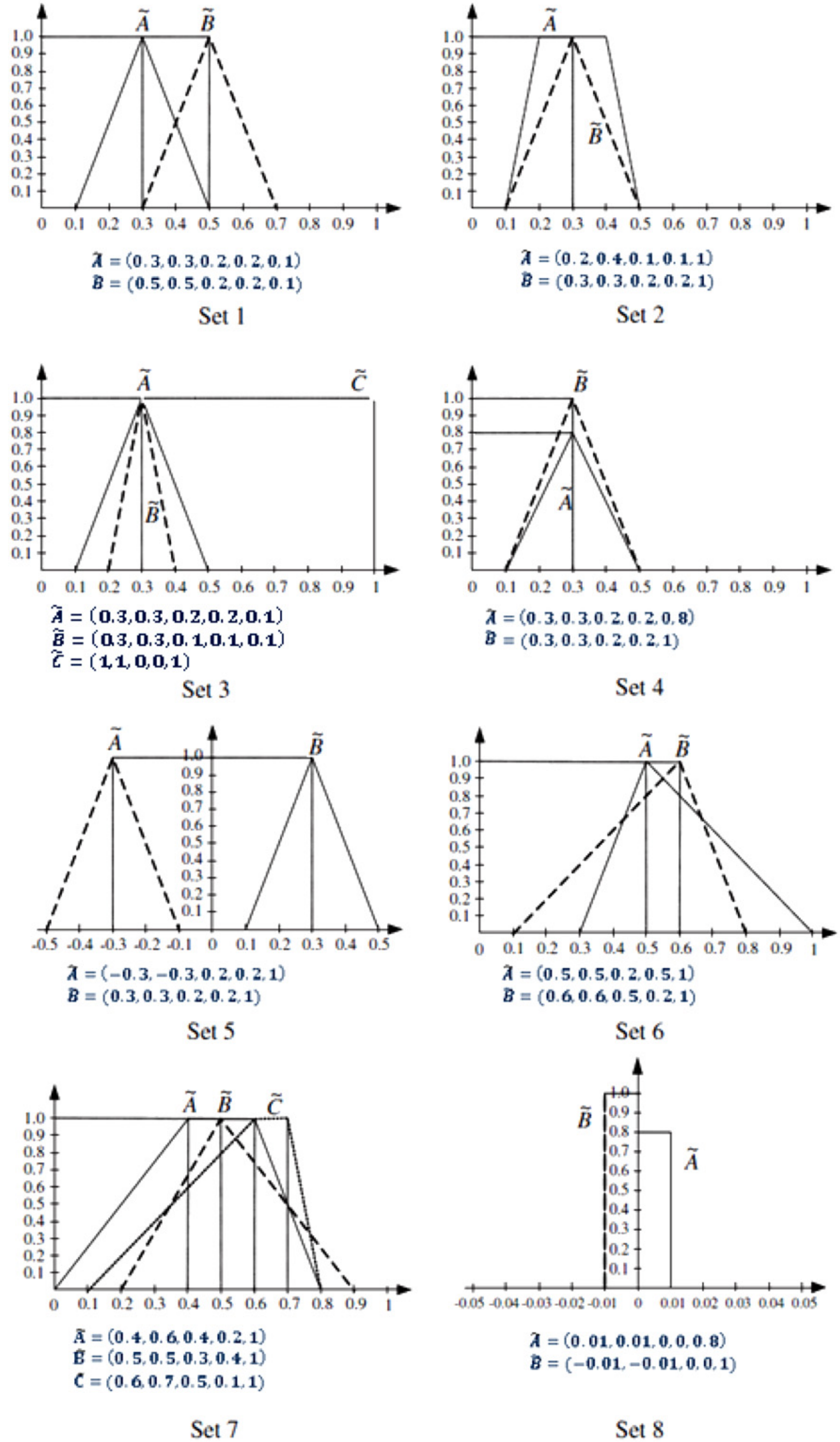

Fig. 5. Eight sets of fuzzy numbers [4].

Table 1
Ranking fuzzy numbers in Example 1.

Methods	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\tilde{A}$	$\tilde{B}$
	Set 1		Set 2		Set 3			Set 4	
Cheng's method	0.5831	0.7071	0.5831	0.5831	0.5831	0.5831	[a]	0.461	0.5831
Chu and Tsao's method	0.15	0.25	0.15	0.15	0.15	0.15	[a]	0.12	0.15
Murakami et al.'s method	0.3	0.5	0.3	0.4167	0.3	0.3	[a]	0.2333	0.3
Yager's method	0.3	0.5	0.3	0.3	0.3	0.3	[a]	0.3	0.3
Chen and Chen's method	0.2579	0.4298	0.2537	0.2597	0.2579	0.2774	1	0.2063	0.2579
Chen and Sanguansat's method	0.3	0.5	0.3	0.3	0.3	0.3	1	0.2824	0.3
Proposed method									
$\alpha = 0.2$	0.48	0.8	0.48	0.48	0.48	0.48	1.6	0.36	0.48
$\alpha = 0.8$	0.12	0.2	0.12	0.12	0.12	0.12	0.4	0	0.12
	Set 5		Set 6		Set 7			Set 8	
Cheng's method	0.5831	0.5831	0.7673	0.7241	0.68	0.7257	0.7462	[a]	[a]
Chu and Tsao's method	−0.15	0.15	0.287	0.2619	0.2281	0.2624	0.2784	[a]	[a]
Murakami et al.'s method	−0.3	0.3	0.6	0.5	0.44	0.5333	0.525	[a]	[a]
Yager's method	−0.3	0.3	0.6	0.5	0.44	0.5333	0.525	[a]	[a]
Chen and Chen's method	−0.2597	0.2579	0.4428	0.4043	0.3354	0.4079	0.4196	0.008	−0.01
Chen and Sanguansat's method	−0.3	0.3	0.575	0.525	0.45	0.525	0.55	0.0094	−0.01
Proposed method									
$\alpha = 0.2$	−0.48	0.48	0.896	0.864	0.736	0.832	0.912	0.012	−0.016
$\alpha = 0.8$	−0.12	0.12	0.206	0.234	0.196	0.202	0.252	0	−0.004

Note: "▮ " means, the unreasonable results in that method.
[a] Denotes the method cannot calculate the ranking value of the fuzzy numbers.

From Table 1, we can see that:

(1) In set 1, all the mentioned methods and the proposed method yield the same ranking order, i.e. $\tilde{A} \prec \tilde{B}$.
(2) In set 2, $\tilde{A} = \tilde{B}$ but Murakami et al.'s method and Chen and Chen's method get unreasonable results.
(3) In set 3, Cheng's method, Chu and Tsao's method, Murakami et al.'s method and Yager's method cannot calculate the ranking score of the crisp-value fuzzy number $\tilde{C}$, whereas Chen and Sanguansat's method and the proposed method can calculate the ranking scores and get the correct ranking order. Moreover, Chen and Chen's method gets unreasonable results.
(4) In set 4, the Yager's method gets unreasonable results but the other methods and the proposed method get the same results.
(5) In set 5, Cheng's method gets unreasonable results in comparing positive and negative fuzzy numbers but the other methods and the proposed method get the same result, i.e. $\tilde{A} \prec \tilde{B}$.
(6) In set 6, all the mentioned methods yield $\tilde{A} \succ \tilde{B}$ but this is wrong, because $\tilde{A}$ and $\tilde{B}$ are two intersected fuzzy numbers and as shown in set 6 of Fig. 3, a different ranking order is obtained when different values of a are taken. For example, if $\alpha \geq 0.8$ then $\tilde{A} \prec_{\alpha} \tilde{B}$.
Thus in this case, the proposed method calculates the correct ranking order, whereas the other methods get unreasonable results.
(7) In set 7, the Yager's method and Murakami et al.'s method get unreasonable results but the other methods and the proposed method get the same results.
(8) In set 8 similar to set 3, Cheng's method, Chu and Tsao's method, Murakami et al.'s method and Yager's method cannot rank crisp-value fuzzy numbers $\tilde{A}$ and $\tilde{B}$, whereas Chen and

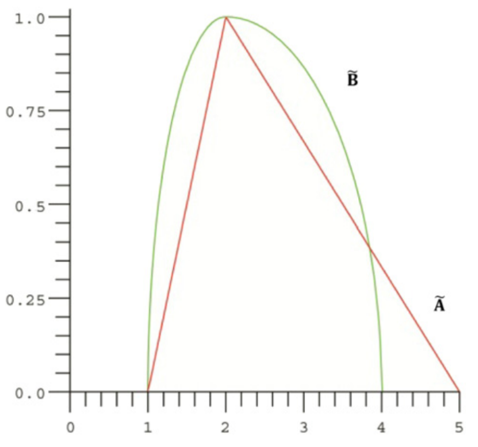

Fig. 6. Fuzzy number in Example 2 [27].

Table 2
Ranking fuzzy numbers in Example 2.

α	$\widetilde{A}$	$\widetilde{B}$	Ranking
0.1	4.41	4.286	$\tilde{A} \succ \tilde{B}$
0.2	3.84	3.787	$\tilde{A} \succ \tilde{B}$
0.8	0.84	0.882	$\tilde{A} \prec \tilde{B}$
0.9	0.41	0.429	$\tilde{A} \prec \tilde{B}$

Sanguansat's method, Chen and Chen's method and the proposed method can calculate the ranking scores and get the correct ranking order.

As seen in the Example 1, the proposed method in this paper can calculate the ranking scores in all of the above sets. Moreover, our computation procedure size is simpler than others.

Example 2. Consider two fuzzy numbers from Ref. [26], i.e. $\tilde{A} = (2, 2, 1, 3, 1)$ that is a triangular fuzzy number and $\tilde{B} = (\underline{B}(r), \overline{B}(r)) = \left(2 - \sqrt{1-r^2},\ 2 + 2\sqrt{1-r^2}\right),\ 0 \le r \le 1$ that is a general fuzzy number, as shown in Fig. 6.

Based on the proposed method we calculate:

$$Q_\alpha(\tilde{A}) = 4(1-\alpha) + (1-\alpha)^2$$

$$\begin{aligned} Q_\alpha(\tilde{B}) &= \int_\alpha^1 \{\underline{B}(r) + \overline{B}(r)\}dr \\ &= \int_\alpha^1 (4 + \sqrt{1-r^2})dr = 4 + \frac{\pi}{4} - \left(4\alpha + \frac{\alpha}{2}\sqrt{1-\alpha^2} + \frac{1}{2}\sin^{-1}\alpha\right). \end{aligned}$$

As shown in the Table 2, for the two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, different ranking orders are obtained when different values of α are taken. For example, if $\alpha = 0.8$ we have $\tilde{A} \prec \tilde{B}$ and if $\alpha = 0.1$, $\tilde{A} \succ \tilde{B}$.

But in Chu and Tsao's method [7] and the method of Wang and Lee [26], we have always $\tilde{A} \succ \tilde{B}$, and by the method of Deng et al. [8], we have always $\tilde{A} \prec \tilde{B}$.

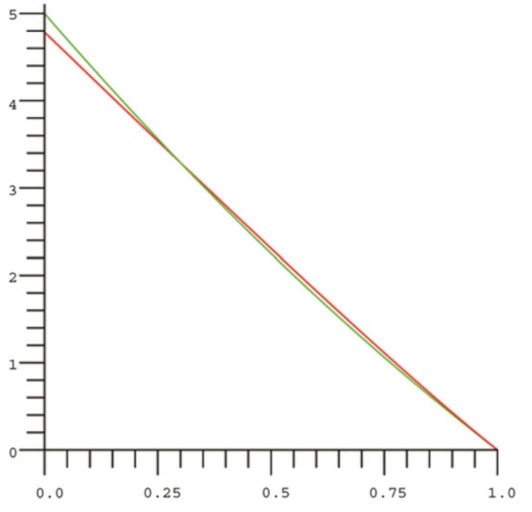

Fig. 7. $Q_\alpha(\widetilde{A})$ (Green color) and $Q_\alpha(\widetilde{B})$ (Red color) functions in Example 2. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

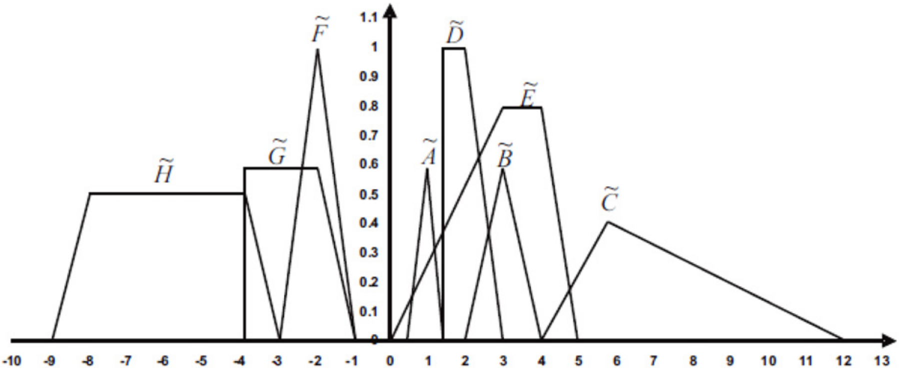

Fig. 8. Eight different fuzzy numbers.

In Fig. 6, we see that $\tilde{A}$ and $\tilde{B}$ are two intersected fuzzy numbers and by plot $Q_\alpha(\tilde{A})$ and $Q_\alpha(\tilde{B})$ in Fig. 7, we see that the proposed method calculates the correct ranking order, whereas the other methods get unreasonable results.

Example 3. In this example, we show the potential of the proposed method to compare many different fuzzy numbers.

From Fig. 8, we can see that:

$$\begin{aligned}
&\tilde{A} = (1, 1, 0.5, 0.5, 0.6), && \tilde{B} = (3, 3, 1, 1, 0.6),\\
&\tilde{C} = (6, 6, 2, 6, 0.4), && \tilde{D} = (1.5, 2, 0, 1, 1)\\
&\tilde{E} = (3, 4, 3, 1, 0.8), && \tilde{F} = (-2, -2, 1, 1, 1),\\
&\tilde{G} = (-4, -2, 0, 1, 0.6), && \tilde{H} = (-8, -4, 1, 1, 0.5).
\end{aligned}$$

Table 3
Ranking fuzzy numbers in Example 3.

α	Ranking
0.1	$\tilde{H} \prec \tilde{F} \prec \tilde{G} \prec \tilde{A} \prec \tilde{B} \prec \tilde{D} \prec \tilde{C} \prec \check{E}$
0.2	$\tilde{H} \prec \tilde{F} \prec \tilde{G} \prec \tilde{A} \prec \tilde{B} \prec \tilde{D} \prec \tilde{C} \prec \check{E}$
0.5	$\tilde{F} \prec \tilde{G} \prec \tilde{H} = \tilde{C} \prec \tilde{A} \prec \tilde{B} \prec \tilde{D} \prec \check{E}$
0.6	$\tilde{F} \prec \tilde{G} = \tilde{H} = \tilde{C} = \tilde{A} = \tilde{B} \prec \tilde{E} \prec \check{D}$
0.8	$\tilde{F} \prec \tilde{G} = \tilde{H} = \tilde{C} = \tilde{A} = \tilde{B} = \tilde{E} \prec \check{D}$
0.9	$\tilde{F} \prec \tilde{G} = \tilde{H} = \tilde{C} = \tilde{A} = \tilde{B} = \tilde{E} \prec \check{D}$

The parametric values corresponding to the above fuzzy numbers are as follows:

$$Q_\alpha\big(\tilde{A}\big) = 2(0.6-\alpha), \qquad Q_\alpha\big(\tilde{B}\big) = 6(0.6-\alpha)$$
$$Q_\alpha\big(\tilde{C}\big) = 12\,(0.4-\alpha) + 5(0.4-\alpha)^2, \qquad Q_\alpha\big(\tilde{D}\big) = 3.5\,(1-\alpha) + 0.5(1-\alpha)^2$$
$$Q_\alpha\big(\tilde{E}\big) = 7\,(0.8-\alpha) - 1.25(0.8-\alpha)^2, \qquad Q_\alpha\big(\tilde{F}\big) = -4(1-\alpha)$$
$$Q_\alpha\big(\tilde{G}\big) = -6\,(0.6-\alpha) + 0.83(0.6-\alpha)^2, \qquad Q_\alpha\big(\tilde{H}\big) = -12(0.5-\alpha).$$

The results of ranking for different decision level (different α value) are shown in Table 3. As we see in Table 3, the ranking order varies when different decision levels are adopted.

5. Conclusion

In decision analysis in a fuzzy environment, ranking fuzzy numbers is a very important decision-making procedure. This paper presents a new approach for ranking fuzzy numbers. The proposed method can rank any kinds of fuzzy numbers with different kinds of membership functions. The proposed method, in this paper, is a complete and very strong method in ranking fuzzy numbers.

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